G2S3: Quantum Linear Algebra **August 11, 2023**

Lecture 2: Quantum Numerical Linear Algebra

Resources: IPAM:

Mathematical and Computational Challenges in Quantum Computing, Long Program

(https://www.ipam.ucla.edu/programs/long-programs/mathematical-and-computational-challenges-in-quantum-computing/?tab=activities)

Quantum Numerical Linear Algebra, Workshop

(https://www.ipam.ucla.edu/programs/workshops/quantum-numerical-linear-algebra/?tab=schedule)

Question: What are the fundamental components of linear algebra?

2.1 Introduction

In classical computing, solving a linear system involves matrix-vector multiplication and various algorithms like Gaussian elimination or LU decomposition.

The Quantum Linear System Algorithm (QLSA) is designed to leverage the properties of quantum computing for solving linear systems:

Given a matrix A and a vector b, the goal is to find a quantum state $|\tilde{x}\rangle$ that encodes the solution x of $Ax = b$.

2.2 Quantum Linear System Problem (QLSP)

Example: $A = \frac{1}{4}X + \frac{3}{4}I$, $|b\rangle = |0\rangle$

2.3 Review: What is a Quantum Computer?

Quantum State Vectors

A quantum state is represented by a complex vector in a Hilbert space. For example, a qubit state can be written as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where α and β are complex probability amplitudes, and $|0\rangle$ and $|1\rangle$ are the basis states.

Pauli Matrices

Used for quantum state manipulation, qubit rotations, and creating entanglement.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hadamard Gate

The Hadamard gate creates superposition:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Quantum Superposition

Using the Hadamard gate, we can transform basis states, and create a quantum superposition:

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

The qubits are simultaneously in both states $|0\rangle$ and $|1\rangle$.

2.3.1 Tensor Product of Basis States

The tensor product is used to describe composite quantum systems.

Tensor Product

The tensor product combines two quantum states to describe their joint state. For example:

$$|0\rangle \otimes |0\rangle = |00\rangle$$

This represents the joint state of two qubits, where the first qubit is in state $|0\rangle$ and the second qubit is in state $|0\rangle$.

Basis States

Consider the basis states for two qubits:

$$|00\rangle = \qquad |01\rangle = \qquad |10\rangle = \qquad |11\rangle =$$

Quantum States

Using the tensor product, we can represent the combined states of two qubits:

$$|00\rangle = |0\rangle \otimes |0\rangle |01\rangle = |0\rangle \otimes |1\rangle |10\rangle = |1\rangle \otimes |0\rangle |11\rangle = |1\rangle \otimes |1\rangle$$

The CNOT gate is a fundamental two-qubit gate in quantum computing:

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

2.4 Block Encoding

Example:

2.4.1 Block Encoding A^{-1}

Polynomials and Matrices:

Characteristic Equation for A:

$p(A)$:

Taylor Expansion of e^A:

Polynomial Approximation of A^{-1}/α:

Constructing $U_{A^{-1}}|b\rangle$

2.5 Quantum Iterative Solvers

Complexity of Conjugate Gradient:

$$O(N\sqrt{\kappa(A)}\log(1/\epsilon)) \tag{2.1}$$

Lower Bound: There is a tradeoff between the condition number of A, the desired accuracy of the solution, and the number of quantum operations required. These operations scale logarithmically with the condition number and high accuracy can require a significant number of quantum operations.

Goal: Achieve a near-optimal quantum linear solver with $\tilde{O}(\kappa(A)\text{polylog}(1/\epsilon))$.

Algorithm	**Complexity**
Quantum Phase Estimation (HHL)	$\tilde{O}\left(\frac{\kappa^2}{\epsilon}\right)$
Linear Combination of Unitaries (LCU)	$\tilde{O}(\kappa^2\text{polylog}(1/\epsilon)$
Quantum Singular Value Transformation (QSVT)	$\tilde{O}(\kappa^2(A)\log(1/\epsilon))$.
Time-Optimal Adiabatic Quantum Computing (AQC(exp))	$\tilde{O}(\kappa\text{polylog}(1/\epsilon))$
Eigenstate Filtering	$\tilde{O}(\kappa\log(1/\epsilon)$

Table 2.1:

2.6 Adiabatic Computation

If a physical system starts in a ground state and evolves slow, it will remain close to the instantaneous ground state. $\Leftrightarrow$ *A physical system remains in its instantaneous eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the Hamiltonian's spectrum.*

Review: Eigendecomposition

2.6.1 Reformulating QLSP as an Eigenvalue Problem

Let's do it classically...

Consider a linear system $A\mathbf{x} = \mathbf{b}$, where A is a Hermitian matrix.

Introduce another vector $\mathbf{y}$ and consider the following transformation:

$$(A - \lambda I)\mathbf{y} = \mathbf{b}$$

where λ is an eigenvalue of matrix A, and I is the identity matrix. Now let $B = A - \lambda I$, and we have

$$B\mathbf{x} = \mathbf{y}$$

Rewriting in "quantum":

QLSP $\rightarrow$ **Eigenproblem**: Find an eigenvector of H with eigenvalue 0.

2.7 Preconditioned Quantum Linear System Solver

2.7.1 Fast Inversion of Diagonal Matrices

2.7.2 Sparse Approximate Inverses

$$\min \|MA - I\|_F^2 = \sum_{k=1}^{N} \|(MA - I)e_k\|_2^2$$

N **Independent Least Squares Problems**:

$$\min_{m_k} \sum_{k=1}^{N} \|Am_k - e_k\|_2^2$$

See: *Preconditioned Quantum Linear Systems Algorithm* (Clader, Jacobs, Sprouse 2013)

2.8 Quantum SVD

Review: (Classical) SVD

2.8.1 Entanglement and the SVD

Entanglement: Two or more quantum systems become correlated such that one cannot be described independently of the other.

Relationship to the SVD: We can think about U and V being correlated via Σ, and measured by the number of nonzero singular values.

Conclusion: ψ is *entangled* if its Schmidt rank is strictly greater than 1 (else, it is *not entangled*).